

Permutations and Primes

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The problem of digital sets (DS) *all* permutations of which generate primes is discussed. Such sets may include only four types of digits: 1, 3, 7 and 9. The direct computations show that for N-digit (ND) integers such DS's occur at $N = 1, 2, 3$, and are absent in the 4 - 10 interval of N . On the other hand in $N = 19, 23, 317$ and 1031 cases (as well as in $N = 2$ case) the formal "total" permutation is provided by "repunits", integers with all digits being 1. The existence/nonexistence of other (not repunits) full-permutation DS's for arbitrary large N is an open question with probable negative answer. Maximal-permutation DS's with maximal number of primes are given for various numbers of digits.

Keywords: Number theory, Permutations, Prime numbers

The primes are real pearls among all integers and are subject of long-standing (several thousand year) researches. The many amazing results are established on primes [1, 2, 3]. The number of primes are infinite (by the way the largest known prime with 2 098 960 digits, $2^{6972593} - 1$, was found quite recently [1]) and their general behavior with n is at present well known and is described, e.g., by some built-in functions of MATHEMATICA [4]: PrimeQ[n] gives True if n is prime and False otherwise, Prime[n] gives n th prime (with assumption Prime[1] = 2), and PrimePi[n] gives number of all primes $\leq n$. Using mainly MATHEMATICA's means I discuss here a problem of the permutations of given digital set all giving prime. Consider some examples first. Already for two-digit primes one may note pairs 13/31, 17/71, 37/73, 79/97 - all primes, (one may add here fifth "pair" 11/11 as well), while 19 "loses" its partner, 91 which is not a prime, see Table 2. One may wonder: is it possible, in the more-digit cases, that full permutations of some DS all give primes?

In 3D case (Table 2, case B) the answer is positive, there are 3 full permutation DS's: 113, 199 and 337, that is, for instance, all three possible permutations 113, 131 and 331 give primes. Other two interesting DS's are 179 and 379 each giving even more primes, 4, which is however less than a corresponding total number of possible permutations - 6. Another DS with 6 permutations, 137, gives only 3 primes.

Notice that in this paper the "basic" DS's are written as first permutations in lexicographic order and they themselves are not necessarily primes. Two smallest such sets are 119 and 133 which are not primes while two other members of each permutation family (191, 911 and 313, 331, respectively) are primes. (And the largest such DS considered in this note is $7_2 9_8 = 7799999999$, see Table 8.)

Here is the place to clarify the point of terminology: speaking about, e.g. 137 as DS, I consider it as what in MATHEMATICA is *List*{1, 3, 7} with 3 elements; at another hand, 137 is also used as 3-digit integer in our usual 10-base arithmetic system. To avoid cumbersome notations I assume this not-too-rigorous approach.

It is evident that *full* permutation may give *all* primes for DS's comprising *only* of digits

1, 3, 7 and 9, and *none* of 0, 2, 4, 5, 6, 8 digits. This greatly reduces the field of search, removing all integers containing at least one of the digits 0, 2, 4, 5, 6 or 8 .

The program was written in MATHEMATICA for searching such full permutations among "1-3-7-9" primes with the negative answer for 4 to 10-digit integers. In all cases number of primes provided by any given DS is *less* than number of *all* possible permutations, see attached Figures and Tables. This is the main (and negative) result, but some other remarks may be made on the by-products of calculations.

1. Primes among all "1-3-7-9" integers may be briefly described by "N-P-D" code as follows:

1 - 2 - 2, 2 - 10 - 6, 3 - 30 - 12, 4 - 63 - 14, 5 - 249 - 35, 6 - 757 - 54, 7 - 2 709 - 74, 8 - 9 177 - 101, 9 - 33 191 - 142, 10 - 118 912 - 184,

where N is number of digits, P is number of primes and D is number of DS's.

For all primes (with arbitrary digits) "N-P-D" figures are:

1 - 4 - 4, 2 - 21 - 17, 3 - 143 - 86, 4 - 1 061 - 336, 5 - 8 363 - 1 109, 6 - 68 906 - 2 967, 7 - 586 081 - 7 041, 8 - 5 096 876 - 15 259, 9 - 45 086 079 - ?, 10 - 404 204 977 - ?.

There is of some interest that for smaller N "mean productivity" (P/D) of "1-3-7-9" DS's are larger than that of all DS's, while starting from N=5 number of primes per DS is larger for all DS:

Table 1. Number of N digit primes per basic DS as function of N for "1-3-7-9" DS's and all DS's

=====		
N	P/D	
=====		
	"1379"	all
=====		
1	1.0000	1.0000
2	1.6667	1.2353
3	2.5000	1.6628
4	4.5000	3.1577
5	7.1143	7.5410
6	14.0185	23.2241
7	36.6081	83.2383
8	90.8614	334.0242
9	233.7394	?
10	646.2609	?

I do not present values of D and distribution of number of primes per given DS in general case by two reasons, first due to necessity of the too time-consuming calculations, and second due to ambiguity of the problem, e.g. what to do with primes with zeros, as in this case some permutation of corresponding DS, with zeros at the beginning, give integers (and possible primes) with less number digits. First three such primes are 101, 103, 107 and last three such 10 digit primes are 9₇701, 9₇703 and 9₇707.

Another observation is that while there is a very strong correlation between p and c, the more rich permutation family does not necessarily give more primes; cf. 139 vs 113, or 1139 vs 1379.

3. As among primes with 4 to 10 digits there is no single full permutation, it is interesting to look for *maximal* permutations. Quite surprisingly, these "maximal" DS's are not at all among "1-3-7-9" DS's. From Table 2 one may note, that maximal "1-3-7-9" DS's with 4 digits are 1379 with 7 primes and 1139 with 8 primes. At the same time, there are 4 "ordinary" DS's (1349, 1457, 3479, 3679) with 9 primes, two (1579 and 1789) with 10, and two "super-sets" 1237 and 1279 with 11 primes. To be able to give more primes DS should comprise the even digits as well, not only odd ones. This fact is really very surprising, because number distribution of digits in, for example, 4D primes is not quite at all random, with great favor to digits 1, 3, 7, 9:

Count[n] = 217, 603, 359, 602, 326, 327, 336, 574, 321, 579, at n=0 to 9, each digit from "1-3-7-9" family is roughly two times more common than any one of other family "0-2-4-5-6-8". In spite of this, DS's of "mixed races" are more productive in making primes. A very interesting observation which may find its application in the fields very far from primes and integers.

4. Some maximal permutation "1379" DS's for the larger numbers of digits are:

	b	c	p	c/p
	=====			
5D	11339	15	30	.5000
	13379	18	60	.3000
	13799	29	60	.4833
6D	113779	60	180	.3333
	133379	35	120	.2917
	133799	55	180	.3056
7D	1113799	113	420	.2690
	1133779	182	630	.2889
	1137799	169	630	.2683
8D	11333779	419	1680	.2494
	11337779	403	1680	.2399
	11377999	397	1680	.2363
9D	113337799	1388	7560	.1836
	111337799	1525	7560	.2017
	113377999	1550	7560	.2050
10D	1113337799	4555	25200	.1808
	1133777999	4606	25200	.1828
	1133377799	4384	25200	.1740

In all cases $c[i] < p[i]$, and relation c/p has a tendency of decreasing with increasing N, that is in some sense, "probability" of appearing of full permutation among "1379" primes diminishes with increasing number of digits. One may consider it as a hint that the full-permutation "1379" DS is absent for arbitrary large N.

I conclude with guess that the existence/nonexistence of (not repunits) DS's for arbitrary large N is an open question with probable negative answer.

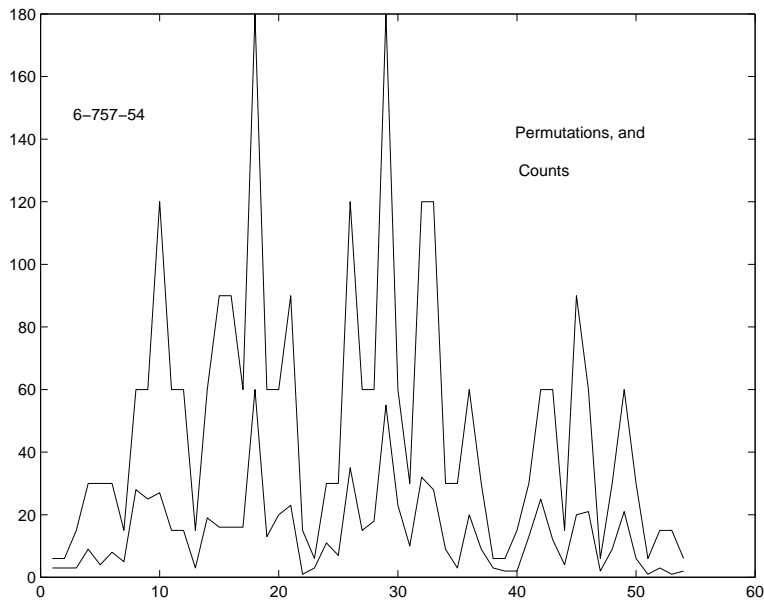


FIG. 1. 6 digit case. Abscissae - order number of basic digit sets (BDS); see Table 4. Ordinates - upper curve: number of permutations of BDS as function of order number of BDS; lower curve: number of primes given by BDS as function of order number of BDS.

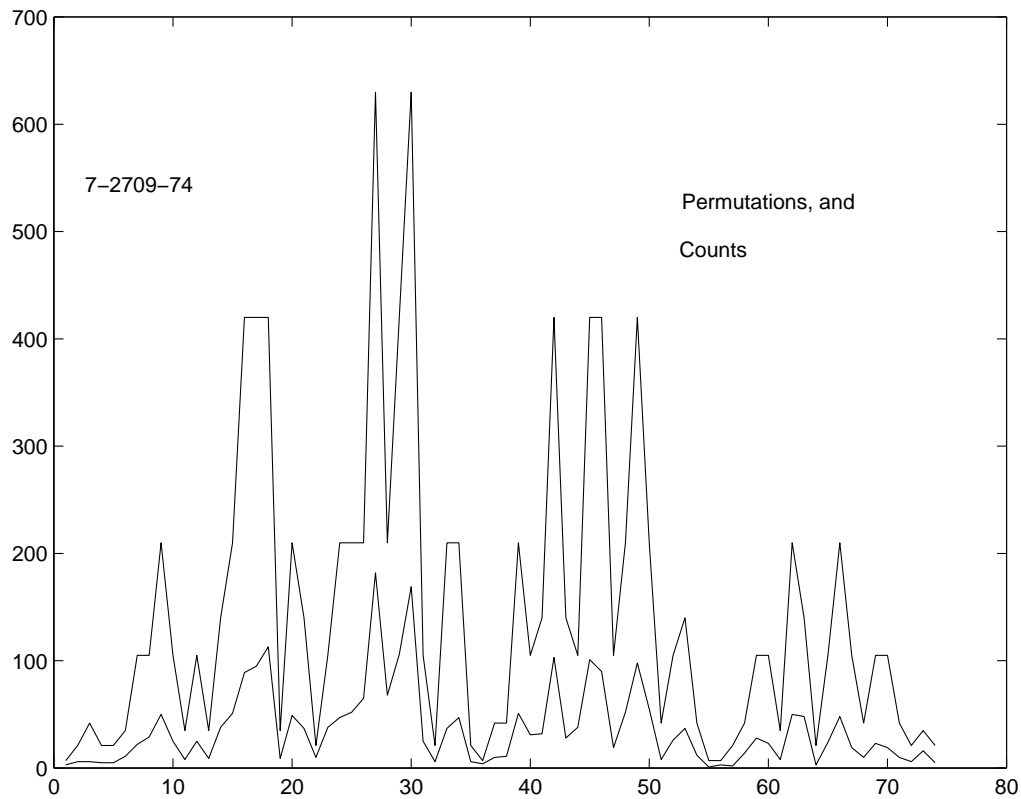


FIG. 2. 7 digit case. Abscissae - order number of basic digit sets (BDS); see Table 5. Ordinates - upper curve: number of permutations of BDS as function of order number of BDS; lower curve: number of primes given by BDS as function of order number of BDS.

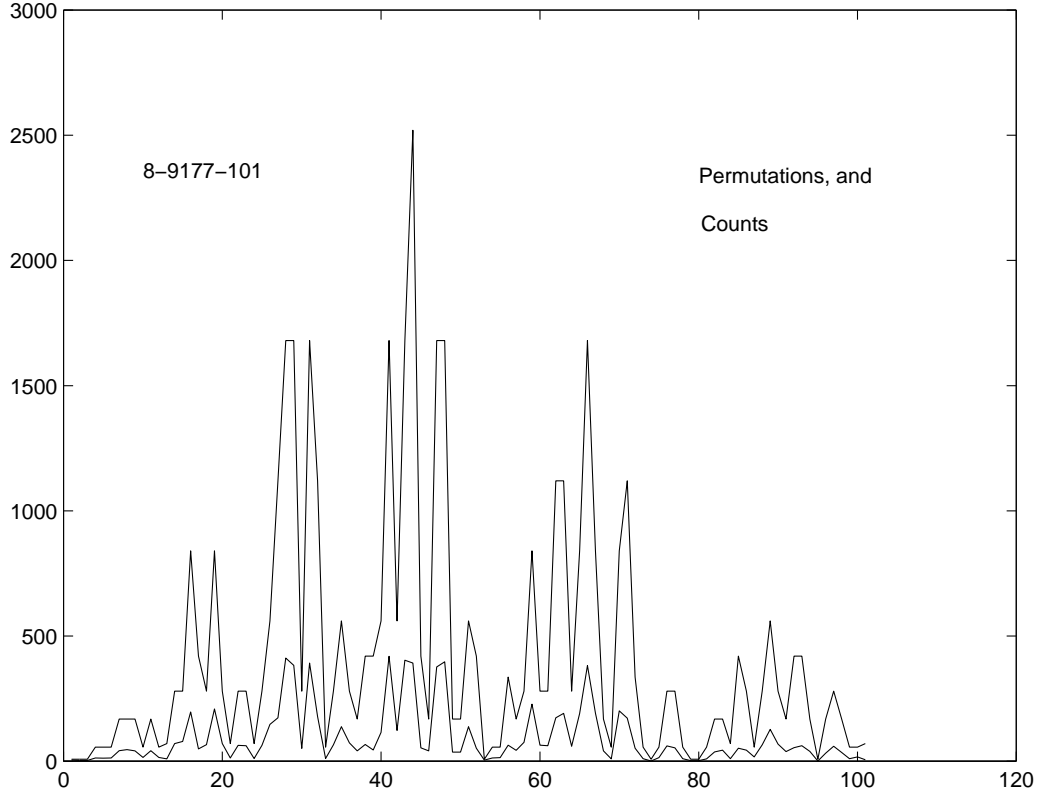


FIG. 3. 8 digit case. Abscissae - order number of basic digit sets (BDS); see Table 6. Ordinates - upper curve: number of permutations of BDS as function of order number of BDS; lower curve: number of primes given by BDS as function of order number of BDS.

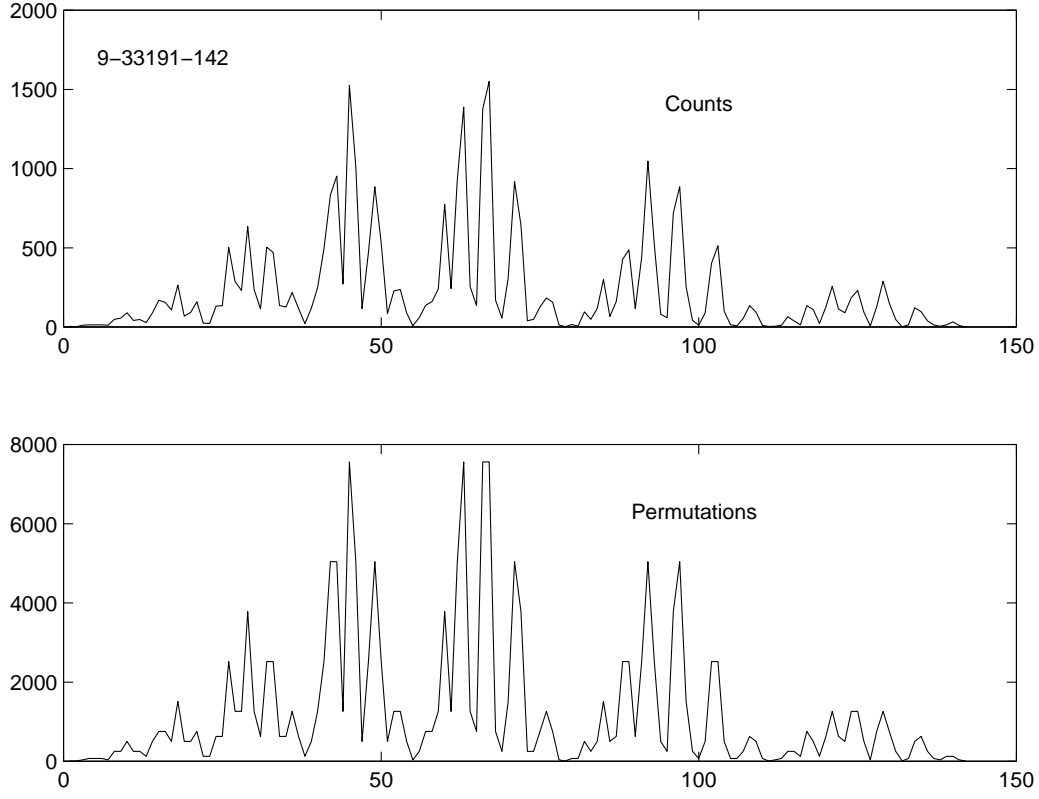


FIG. 4. 9 digit case. Abscissae - order number of basic digit sets (BDS); see Table 7. Ordinates - upper panel: number of primes given by BDS as function of order number of lower panel: number of permutations of BDS as function of order number of BDS

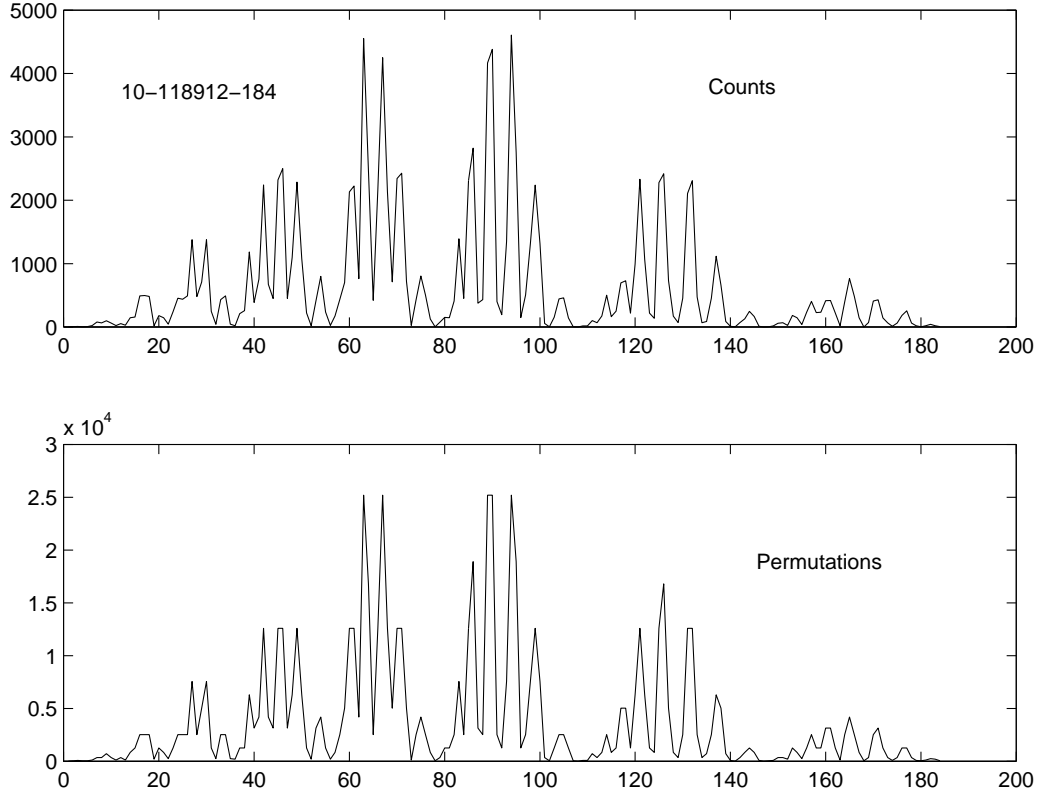


FIG. 5. 10 digit case. Abscissae - order number of basic digit sets (BDS); see Table 8. Ordinates - upper panel: number of primes given by BDS as function of order number of lower panel: number of permutations of BDS as function of order number of BDS

Table 2. A: 2-10-6 case. b - basic digit sets,
c - counts, p - permutations.
Total number 10 = Sum[c[i]].
B: 3-30-12 case.
c[i] = p[i] at 3 cases, for b=113, 199 and 337.
C: 4-63-14 case.
c[i] < p[i] at all cases.
Note that basic digit sets are written as first
ones in lexicografic order and they are not
necessary primes themselves. Two least such sets are
119 and 133 which are not primes while two other
members of each permutation family (191, 911 and
313, 331, respectively) are primes.
Another observation is that while there is a very
strong correlation between p and c, the more rich
permutation family does not necessarily
give more primes; cf. 139 vs 113, or 1139 vs 1379

A			B			C		
b	c	p	b	c	p	b	c	p
11	1	1	113	3	3	1117	2	4
13	2	2	119	2	3	1139	8	12
17	2	2	133	2	3	1333	2	4
19	1	2	137	3	6	1337	5	12
37	2	2	139	2	6	1339	5	12
79	2	2	179	4	6	1379	7	24
-----			199	3	3	1399	6	12
10			337	3	3	1777	3	4
			377	1	3	1799	5	12
			379	4	6	1999	2	4
			779	2	3	3337	3	4
			799	1	3	3379	6	12
			-----			3779	5	12
			30			3799	4	12

						63		

Table 3. 5-249-35 case. b - 35 basic digit sets,
c - counts, p - permutations.
Total number 249 = Sum[c[i]].
c[i] < p[i] at all i.

b	c	p	b	c	p	b	c	p
=====								
11113	3	5	11117	2	5	11119	1	5
11137	5	20	11177	5	10	11179	9	20
11333	4	10	11339	15	30	11377	10	30
11399	13	30	11777	3	10	11779	12	30
11999	5	10	13333	2	5	13337	9	20
13339	9	20	13379	18	60	13399	8	30
13777	8	20	13799	29	60	13999	7	20
17777	1	5	17779	5	20	17999	7	20
19999	1	5	33377	3	10	33379	7	20
33779	10	30	33799	9	30	37777	2	5
37799	10	30	37999	9	20	77779	4	5
77999	3	10	79999	1	5			

249

Table 4. 6-757-54 Case

=====		
b	c	p
=====		
111113	3	6
111119	3	6
111133	3	15
111137	9	30
111139	4	30
111179	8	30
111199	5	15
111337	28	60
111377	25	60
111379	27	120
111779	15	60
111799	15	60
113333	3	15
113339	19	60
113377	16	90
113399	16	90
113777	16	60
113779	60	180
113999	13	60
117779	20	60
117799	23	90
119999	1	15
133333	3	6
133337	11	30
133339	7	30
133379	35	120
133399	15	60
133777	18	60
133799	55	180
133999	23	60
137777	10	30
137779	32	120
137999	28	120
139999	9	30
177779	3	30
177799	20	60
179999	9	30
199999	3	6
333337	2	6
333377	2	15
333379	13	30

333779	25	60
333799	12	60
337777	4	15
337799	20	90
337999	21	60
377777	2	6
377779	9	30
377999	21	60
379999	6	30
777779	1	6
777799	3	15
779999	1	15
799999	2	6

Table 5. 7-2709-74 Case

=====		
b	c	p
=====		
1111117	3	7
1111133	6	21
1111139	6	42
1111177	5	21
1111199	5	21
1111333	11	35
1111337	22	105
1111339	29	105
1111379	50	210
1111399	25	105
1111777	8	35
1111799	25	105
1111999	9	35
1113337	38	140
1113377	51	210
1113379	89	420
1113779	95	420
1113799	113	420
1117777	9	35
1117799	49	210
1117999	37	140
1133333	10	21
1133339	38	105
1133377	47	210
1133399	52	210
1133777	65	210
1133779	182	630
1133999	68	210
1137779	106	420
1137799	169	630
1139999	25	105
1177777	6	21
1177799	37	210
1177999	47	210
1199999	6	21
1333333	4	7
1333337	10	42
1333339	11	42
1333379	51	210
1333399	31	105
1333777	32	140

1333799	103	420
1333999	28	140
1337777	38	105
1337779	101	420
1337999	90	420
1339999	19	105
1377779	52	210
1377799	98	420
1379999	55	210
1399999	8	42
1777799	26	105
1777999	37	140
1799999	12	42
1999999	1	7
3333337	3	7
3333377	2	21
3333379	14	42
3333779	28	105
3333799	23	105
3337777	8	35
3337799	50	210
3337999	48	140
3377777	3	21
3377779	24	105
3377999	48	210
3379999	19	105
3777779	10	42
3777799	23	105
3779999	19	105
3799999	10	42
7777799	6	21
7777999	16	35
7799999	5	21

Table 6. 8-9177-101 Case

b	c	p
11111113	4	8
11111117	3	8
11111119	3	8
11111137	13	56
11111179	12	56
11111333	13	56
11111339	42	168
11111377	46	168
11111399	41	168
11111777	15	56
11111779	42	168
11111999	15	56
11113333	9	70
11113337	70	280
11113339	79	280
11113379	197	840
11113399	49	420
11113777	66	280
11113799	209	840
11113999	70	280
11117777	13	70
11117779	63	280
11117999	62	280
11119999	11	70
11133337	63	280
11133377	147	560
11133379	173	1120
11133779	412	1680
11133799	383	1680
11137777	51	280
11137799	392	1680
11137999	178	1120
11177777	11	56
11177779	65	280
11177999	138	560
11179999	73	280
11333339	41	168
11333377	67	420
11333399	45	420
11333777	115	560
11333779	419	1680

11333999	123	560
11337779	403	1680
11337799	392	2520
11339999	53	420
11377777	41	168
11377799	376	1680
11377999	397	1680
11399999	36	168
11777779	36	168
11777999	138	560
11779999	52	420
13333333	4	8
13333337	13	56
13333339	14	56
13333379	64	336
13333399	43	168
13333777	75	280
13333799	228	840
13333999	64	280
13337777	62	280
13337779	173	1120
13337999	191	1120
13339999	60	280
13377779	190	840
13377799	382	1680
13379999	192	840
13399999	41	168
13777777	9	56
13777799	201	840
13777999	172	1120
13799999	52	336
13999999	10	56
17777777	1	8
17777779	14	56
17777999	61	280
17779999	53	280
17999999	10	56
19999999	2	8
33333337	2	8
33333379	9	56
33333779	37	168
33333799	44	168
33337777	10	70
33337799	52	420
33337999	44	280

33377777	17	56
33377779	65	280
33377999	127	560
33379999	68	280
33777779	38	168
33777799	54	420
33779999	62	420
33799999	39	168
37777777	1	8
37777799	32	168
37777999	60	280
37799999	35	168
37999999	10	56
77777999	17	56
77779999	6	70

Table 7. 9-33191-142 Case

b	c	p
111111113	4	9
111111119	1	9
111111133	11	36
111111137	14	72
111111139	14	72
111111179	14	72
111111199	11	36
111111337	48	252
111111377	56	252
111111379	90	504
111111779	41	252
111111799	47	252
111113333	28	126
111113339	89	504
111113377	169	756
111113399	155	756
111113777	108	504
111113779	265	1512
111113999	69	504
111117779	92	504
111117799	159	756
111119999	24	126
111133333	22	126
111133337	131	630
111133339	135	630
111133379	503	2520
111133399	286	1260
111133777	231	1260
111133799	634	3780
111133999	237	1260
111137777	113	630
111137779	504	2520
111137999	470	2520
111139999	136	630
111177779	126	630
111177799	218	1260
111179999	117	630
111199999	21	126
111333337	119	504
111333377	248	1260

111333379	495	2520
111333779	833	5040
111333799	954	5040
111337777	272	1260
111337799	1525	7560
111337999	1012	5040
111377777	116	504
111377779	470	2520
111377999	886	5040
111379999	528	2520
111777779	84	504
111777799	227	1260
111779999	237	1260
111799999	89	504
113333333	8	36
113333339	59	252
113333377	137	756
113333399	161	756
113333777	240	1260
113333779	776	3780
113333999	242	1260
113337779	931	5040
113337799	1388	7560
113339999	255	1260
113377777	136	756
113377799	1376	7560
113377999	1550	7560
113399999	170	756
113777777	56	252
113777779	308	1512
113777999	918	5040
113779999	642	3780
113999999	39	252
117777779	48	252
117777799	127	756
117779999	183	1260
117799999	157	756
119999999	12	36
133333333	2	9
133333337	16	72
133333339	6	72
133333379	95	504
133333399	48	252
133333777	117	504
133333799	300	1512

133333999	66	504
133337777	161	630
133337779	428	2520
133337999	487	2520
133339999	116	630
133377779	440	2520
133377799	1047	5040
133379999	522	2520
133399999	81	504
133777777	58	252
133777799	719	3780
133777999	885	5040
133799999	254	1512
133999999	42	252
137777777	11	72
137777779	89	504
137777999	403	2520
137779999	513	2520
137999999	100	504
139999999	13	72
177777779	8	72
177777799	54	252
177779999	135	630
177799999	94	504
179999999	10	72
333333337	3	9
333333377	5	36
333333379	11	72
333333779	65	252
333333799	38	252
333337777	14	126
333337799	135	756
333337999	110	504
333377777	23	126
333377779	126	630
333377999	257	1260
333379999	113	630
333777779	90	504
333777799	185	1260
333779999	230	1260
333799999	93	504
337777777	7	36
337777799	128	756
337777999	290	1260
337799999	150	756

337999999	47	252
377777777	1	9
377777779	14	72
377777999	121	504
377779999	96	630
377999999	38	252
379999999	14	72
777777799	5	36
777779999	13	126
777799999	32	126
779999999	9	36
799999999	1	9

Table 8. 10-118912-184 Case

=====		
b	c	p
=====		
1111111117	2	10
1111111133	5	45
1111111139	9	90
1111111177	4	45
1111111199	5	45
1111111333	22	120
1111111337	79	360
1111111339	64	360
1111111379	97	720
1111111399	62	360
1111111777	23	120
1111111799	54	360
1111111999	29	120
1111113337	147	840
1111113377	157	1260
1111113379	492	2520
1111113779	496	2520
1111113799	482	2520
1111117777	20	210
1111117799	180	1260
1111117999	147	840
1111133333	45	252
1111133339	241	1260
1111133377	455	2520
1111133399	437	2520
1111133777	491	2520
1111133779	1379	7560
1111133999	479	2520
1111137779	709	5040
1111137799	1379	7560
1111139999	246	1260
1111177777	41	252
1111177799	433	2520
1111177999	490	2520
1111199999	45	252
1111333333	20	210
1111333337	210	1260
1111333339	259	1260
1111333379	1184	6300
1111333399	388	3150
1111333777	754	4200

1111333799	2242	12600
1111333999	673	4200
1111337777	449	3150
1111337779	2320	12600
1111337999	2500	12600
1111339999	451	3150
1111377779	1096	6300
1111377799	2288	12600
1111379999	1077	6300
1111399999	222	1260
1111777777	20	210
1111777799	414	3150
1111777999	799	4200
1111799999	235	1260
1111999999	24	210
1113333337	174	840
1113333377	435	2520
1113333379	702	5040
1113333779	2132	12600
1113333799	2225	12600
1113337777	763	4200
1113337799	4555	25200
1113337999	2473	16800
1113377777	421	2520
1113377779	2171	12600
1113377999	4252	25200
1113379999	2131	12600
1113777779	713	5040
1113777799	2343	12600
1113779999	2426	12600
1113799999	721	5040
1117777777	24	120
1117777799	428	2520
1117777999	806	4200
1117799999	503	2520
1117999999	127	840
1133333333	2	45
1133333339	76	360
1133333377	151	1260
1133333399	145	1260
1133333777	410	2520
1133333779	1392	7560
1133333999	453	2520
1133337779	2311	12600
1133337799	2823	18900

1133339999	376	3150
1133377777	431	2520
1133377799	4166	25200
1133377999	4384	25200
1133399999	400	2520
1133777777	194	1260
1133777779	1346	7560
1133777999	4606	25200
1133779999	2755	18900
1133999999	148	1260
1137777779	505	2520
1137777799	1317	7560
1137779999	2240	12600
1137799999	1317	7560
1139999999	56	360
1177777777	2	45
1177777799	153	1260
1177777999	442	2520
1177799999	459	2520
1177999999	142	1260
1199999999	4	45
1333333333	1	10
1333333337	18	90
1333333339	16	90
1333333379	101	720
1333333399	67	360
1333333777	174	840
1333333799	502	2520
1333333999	163	840
1333337777	250	1260
1333337779	697	5040
1333337999	729	5040
1333339999	219	1260
1333377779	993	6300
1333377799	2332	12600
1333379999	1066	6300
1333399999	218	1260
1333777777	137	840
1333777799	2279	12600
1333777999	2419	16800
1333799999	733	5040
1333999999	172	840
1337777777	69	360
1337777779	455	2520
1337777999	2107	12600

1337779999	2310	12600
1337999999	473	2520
1339999999	67	360
1377777779	87	720
1377777799	454	2520
1377779999	1118	6300
1377799999	659	5040
1379999999	86	720
1399999999	12	90
1777777777	2	10
1777777799	70	360
1777777999	130	840
1777799999	247	1260
1777999999	166	840
1799999999	11	90
1999999999	1	10
3333333377	2	45
3333333379	16	90
3333333779	58	360
3333333799	68	360
3333337777	25	210
3333337799	181	1260
3333337999	145	840
3333377777	38	252
3333377779	233	1260
3333377999	405	2520
3333379999	226	1260
3333777779	233	1260
3333777799	415	3150
3333779999	419	3150
3333799999	221	1260
3337777777	21	120
3337777799	395	2520
3337777999	765	4200
3337799999	486	2520
3337999999	147	840
3377777777	1	45
3377777779	70	360
3377777999	409	2520
3377779999	430	3150
3377999999	144	1260
3379999999	65	360
3777777779	12	90
3777777799	62	360
3777779999	182	1260

3777799999	256	1260
3779999999	62	360
3799999999	20	90
7777777799	2	45
7777777999	19	120
7777799999	41	252
7777999999	21	210
7799999999	5	45

[1] www.utm.edu/research/primes.

[2] [www.utm.edu/research/primes/ lists/top_ten/topten36.htm/E9E36](http://www.utm.edu/research/primes/lists/top_ten/topten36.htm/E9E36).

It should be noted that at this web-page 36 "absolute prime numbers" are given all coinsiding with primes given by the full-permutation DS's considered in this note. Moreover the notions are somewhere in [1,2] presented that: a) "all other repunits are composite to 1₃₀₀₀₀" and b)"an absolute prime is one that remains a prime, for all permutations of its digits".

[3] L. Alexandrov, math/9811096.

[[4] S. Wolfram. Mathematica. 2nd ed. Addison Wesley, 1991